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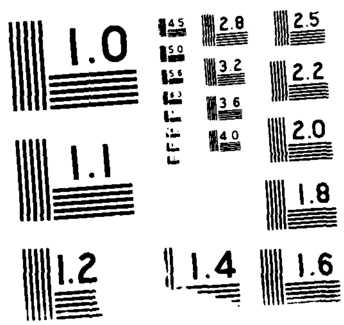
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J. A. Nohel[1], R. C. Rogers[2], and A. E. Tzavaras[3]

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ABSTRACT

We consider a one-dimensional model problem for the motion of a viscoelastic material with fading memory governed by a quasilinear hyperbolic system of integrodifferential equations of Volterra type. For given Cauchy data in $L^\infty(\mathbf{R}) \cap L^2(\mathbf{R})$, we use the method of vanishing viscosity and techniques of compensated compactness to obtain the existence of a weak solution (in the class of bounded measurable functions) in a special case.

AMS (MOS) Subject Classifications: 35B45, 35L60, 35L65, 35K55, 45K05, 73F15

Key Words: viscoelastic materials, fading memory, weak solutions, vanishing viscosity, compensated compactness

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WEAK SOLUTIONS FOR A NONLINEAR SYSTEM IN VISCOELASTICITY

J.A. Nohel[1], R.C. Rogers[2], and A.E. Tzavaras[3]

1. INTRODUCTION.

The purpose of this paper is to initiate a program of investigating the existence of global weak solutions, in the class of bounded measurable functions, for the Cauchy problem

$$(VE)\qquad \begin{cases} w_t = v_x, \\ v_t = \sigma_x, \end{cases} \quad x \in \mathbf{R},\ t > 0, \qquad w(x,0) = w_0(x),\ v(x,0) = v_0(x), \quad x \in \mathbf{R},$$

where the function $\sigma(x,t)$ is determined by the history of $w(x,\cdot)$ through the constitutive assumption

$$(CA)\qquad \sigma(x,t) = \varphi(w(x,t)) + \int_0^t k(t-\tau)\psi(w(x,\tau))d\tau.$$

The given functions $\varphi(w), \psi(w)$ and $k(t)$ are assumed to be smooth and, in addition,

$$\varphi'(w) > 0, \quad w \in \mathbf{R},$$

so that the structure of (VE) is hyperbolic. In this paper the aforementioned goal is achieved in the important special case $\psi \equiv \varphi$, when the data $w_0, v_0 \in L^\infty(\mathbf{R}) \cap L^2(\mathbf{R})$, using the method of vanishing viscosity and techniques of compensated compactness (cf. [21, 22, 23, 7, 17]).

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The system (VE) is a model for one-dimensional motion of a viscoelastic material with fading memory. The functions $v(x,t), w(x,t)$ and $\sigma(x,t)$ stand for the velocity, deformation gradient and stress, respectively, while the constitutive assumption (CA) states that the stress is a particular functional of the history of the deformation gradient; in (CA) the history of $w(x,\cdot)$ is assumed to be zero for $t < 0$ and the body force is taken to be zero. Under appropriate assumptions on the kernel k, the memory induces a weak dissipative mechanism which competes with the hyperbolic character of (VE). Under such assumptions, qualitative properties that would make (VE) a reasonable model for the motion of viscoelastic materials include:

(a) For smooth and small data w_0, v_0, (VE) should possess globally defined classical solutions, which decay to equilibrium as $t \to \infty$. Such results have been established in [12, 5, 20] if $\psi \equiv \varphi$, and in [6, 9] if $\psi \neq \varphi$; for more general models, see [18].

(b) By contrast, if the smooth data are chosen sufficiently large, smooth solutions of (VE) should develop singularities in their derivatives in finite time. This has been shown in [2, 16]; for other models see [18].

(c) For arbitrary L^∞ data w_0, v_0, weak solutions should exist in $\mathbf{R} \times [0;\infty)$. This property is established here in the special case $\psi \equiv \varphi$.

There is a parallel theory for the single conservation law with memory

$$(CLM) \qquad \begin{cases} w_t + \sigma_x = 0, \\ w(x,0) = w_0(x) \end{cases}$$

where $\sigma(x,t)$ is given by (CA). For this equation property (c) was recently established by *Dafermos [4] using the method of compensated compactness; unfortunately, this elegant* approach does not seem to extend to (VE).

In the sequel, we restrict attention to the case $\varphi \equiv \psi$. Then (VE) reads

$$(1.1) \qquad \begin{cases} w_t = v_x, & \\ v_t = \varphi(w)_x + \int_0^t k(t-\tau)\varphi(w(x,\tau))_x d\tau & x \in \mathbf{R},\ t > 0, \\ w(x,0) = w_0(x),\ v(x,0) = v_0(x), & x \in \mathbf{R}. \end{cases}$$

Regarding properties (a), (b) and (c), it is instructive to compare (1.1) with the equations of one-dimensional elasticity

$$(E) \qquad \begin{cases} w_t = v_x \\ v_t = \varphi(w)_x, \end{cases}$$

and with the equations of one-dimensional frictionally damped elastic materials

$$(FE) \qquad \begin{cases} w_t = v_x \\ v_t = \varphi(w)_x - v. \end{cases}$$

The equations of elasticity enjoy property (c) (DiPerna [7]); by contrast, property (a) is not pertinent to (E), since smooth solutions of (E) generally break down in finite time. On

the other hand (FE) exhibits a closer resemblance to (1.1) enjoying all the properties (a), (b) and (c) (see [15,19, 7] respectively).

The similarity between (1.1) and (FE) is revealed by the following transformation due to McCamy [12]. Let $r(t)$ be the resolvent kernel associated with k; i.e., r is the solution of the linear Volterra equation

$$r(t) + \int_0^t k(t-\tau)r(\tau)d\tau = k(t), \quad t \geq 0. \tag{r}$$

Convolving $(1.1)_2$ with $r(t)$, a simple calculation yields

$$\begin{aligned} \int_0^t k(t-\tau)\varphi(w(x,\tau))_x d\tau &= \int_0^t r(t-\tau)v_t(x,\tau)d\tau \\ &= r(0)v(x,t) - r(t)v_0(x) + \int_0^t r'(t-\tau)v(x,\tau)d\tau. \end{aligned} \tag{1.2}$$

Thus, for smooth solutions (1.1) is equivalent to

$$\begin{cases} \left.\begin{aligned} w_t &= v_x, \\ v_t &= \varphi(w)_x + \mathcal{F}[v], \end{aligned}\right\} \quad x \in \mathbf{R},\ t > 0, \\ w(x,0) = w_0(x),\ v(x,0) = v_0(x) \quad x \in \mathbf{R}, \end{cases} \tag{1.3}$$

where

$$\mathcal{F}[v](x,t) := r(0)v(x,t) - r(t)v_0(x) + \int_0^t r'(t-\tau)v(x,\tau)d\tau. \tag{1.4}$$

Under physically reasonable assumptions regarding the kernel k (e.g. $k = a'$, with a smooth, positive, decreasing and convex on $[0,\infty)$), $r(0) = a'(0) < 0$ and the term $r(0)v(x,t)$ has a damping effect. The effect of this term is dominant close to equilibrium thereby inducing property (a). However, far from equilibrium, the hyperbolic part of (1.1) dominates, irrespective of the sign properties of k. It is thus conceivable that the analysis of weak solutions is not affected by the sign properties of the kernel $k(t)$.

We assume: the constitutive function φ satisfies

$$\begin{cases} \varphi : \mathbf{R} \to \mathbf{R} \text{ is a twice continuously differentiable} \\ \text{function such that } \varphi'(w) > 0,\ w \in \mathbf{R}; \\ \varphi \text{ has a single inflection point at } w = w_i \text{ and is} \\ \text{convex on } (w_i,\infty) \text{ and concave on } (-\infty, w_i). \end{cases} \tag{1.5}$$

The kernel k satisfies

$$k : [0,\infty) \to \mathbf{R}, \quad k \in C^1[0,\infty), \tag{1.6}$$

and the data $w_0(x), v_0(x)$ satisfy

$$w_0(x),\ v_0(x) \in L^\infty(\mathbf{R}) \cap L^2(\mathbf{R}). \tag{1.7}$$

We prove:

Theorem 1.1. *Let the hypotheses (1.5)-(1.7) be satisfied. Given $T > 0$, there exists a weak solution $\{u(x,t), v(x,t)\}$ of (1.1) on $\mathbf{R} \times [0,T]$, such that*

$$(w,v) \in L^\infty([0,T]; L^2(\mathbf{R})) \cap L^\infty(\mathbf{R} \times [0,T]).$$

The proof is carried out in Sections 2, 3 and 4 using the method of compensated compactness of Murat [13] and Tartar [21, 22, 23]. This approach has been employed with success by Tartar [22] to obtain L^∞ solutions for the general, scalar Burgers equation, by DiPerna [7] and Rascle [17], to construct L^∞ solutions to the hyperbolic system (E), and by Dafermos [4], to obtain L^∞ solutions for (CLM).

The weak solutions of (1.1) will be sought as the $\varepsilon \searrow 0$ limits of solutions of the regularized problem

$$\begin{cases} w_t = v_x + \varepsilon w_{xx}, & \\ v_t = \varphi(w)_x + \int_0^t k(t-\tau)\varphi(w(x,\tau))_x d\tau & (x,t) \in \mathbf{R} \times (0,T], \\ \quad +\varepsilon(v_{xx} + \int_0^t k(t-\tau)v_{xx}(x,\tau)d\tau), & \\ & \\ w(x,0) = w_0(x),\ v(x,0) = v_0(x), & x \in \mathbf{R}. \end{cases} \tag{1.8}$$

This unconventional regularization preserves one of the main features of (1.1); namely, that the integration with respect to t offsets differentiation with respect to x and the memory terms are of lower order. In Section 2 the system (1.8) is transformed into a parabolic regularization of (1.3). It is also proved there that the initial value problems (1.1) and (1.3) are equivalent for weak solutions.

In Section 3 we prove an existence theorem for (1.8) and obtain the necessary a-priori estimates in order to pass to the limit as $\varepsilon \searrow 0$. Most notably, using a class of exponentially growing convex entropies of (E) constructed by Dafermos [3], we show that the solutions $\{w_\varepsilon, v_\varepsilon\}$ of (1.8) lie in a bounded set in L^∞, uniformly in ε. (In view of the nonlocal nature of the memory term, it does not appear possible to establish L^∞-estimates for solutions of (1.8) by constructing invariant regions.)

The results of Section 3 imply that a subsequence can be extracted converging in L^∞ weak star. Under the L^∞ weak star convergence, composite weak limits are characterized as expected values of a family of probability measures $\nu(x,t)$, called Young measures (Tartar [21, 22, 23]). In Section 4 we employ the techniques of DiPerna [7] concerning the convergence of approximate solutions to the equations of elasticity (E) and a lemma of Murat [14] to show that the Young measure $\nu(x,t)$ reduces to a Dirac mass for almost all (x,t). Thus, a subsequence $\{w_{\varepsilon'}, v_{\varepsilon'}\}$ of solutions of (1.8) converges almost everywhere and the limit is a weak solution of (1.1).

Certain results concerning weak solutions of (VE) have appeared in the literature. Greenberg [8] (also see [18], Section 2.6) established the existence of travelling wave solutions (steady compression shocks) for a history valued problem associated with (VE). Boldrini [1], used the compensated compactness method, to show that as the memory weakens ($k = k(\delta, t) = 0(\delta)$), uniformly bounded solutions $\{w^\delta, v^\delta\}$ of (1.8) converge, as $\delta \downarrow 0$, to a solution of the elastic problem (E).

2. WEAK SOLUTIONS; THE REGULARIZED PROBLEM.

We seek weak solutions of (1.1) in the space L^∞ of bounded measurable functions. Within this framework, a pair of functions $\{w(x,t), v(x,t)\}$ in $L^\infty([0,T]; L^2(\mathbf{R}) \cap L^\infty(\mathbf{R}))$ is a weak solution of (1.1) on $[0,T] \times \mathbf{R}$ if it satisfies

$$\int_0^T \int_{-\infty}^{\infty} (\xi_t w - \xi_x v) dx dt + \int_{-\infty}^{\infty} \xi(x,0) w_0(x) dx = 0, \tag{2.1}$$

$$\begin{aligned} &\int_0^T \int_{-\infty}^{\infty} [\zeta_t v - \zeta_x(\varphi(w) + \int_0^t k(t-\tau)\varphi(w(\cdot,\tau))d\tau)] dx dt \\ &\qquad + \int_{-\infty}^{\infty} \zeta(x,0) v_0(x) dx = 0, \end{aligned} \tag{2.2}$$

for any pair $\{\xi(x,t), \zeta(x,t)\}$ of C^1 functions with compact support on $\mathbf{R} \times [0,T]$ satisfying $\xi(x,T) = \zeta(x,T) = 0$.

The weak solutions of (1.1) will be constructed as the $\varepsilon \searrow 0$ limits of solutions of the regularized system (1.8). The relevance of the unconventional regularization (1.8) is revealed by the following calculation. Convolving $(1.8)_2$ with $r(t)$, the resolvent kernel of $k(t)$, and using (r) and (1.4) we arrive at

$$\begin{aligned} \int_0^t k(t-\tau)(\varphi(w(x,\tau))_x + \varepsilon v_{xx}(x,\tau)) dx d\tau &= \int_0^t r(t-\tau) v_t(x,\tau) d\tau \\ = r(0)v(x,t) - r(t)v_0(x) + \int_0^t r'(t-\tau)v(x,\tau)d\tau &= \mathcal{F}[v](x,t). \end{aligned} \tag{2.3}$$

Thus, the initial value problem (1.8) can be written in the form

$$\begin{cases} w_t = v_x + \varepsilon w_{xx}, \\ v_t = \varphi(w)_x + \mathcal{F}[v] + \varepsilon v_{xx}, \end{cases} \quad (x,t) \in \mathbf{R} \times (0,T] \\ \quad w(x,0) = w_0(x), \; v(x,0) = v_0(x), \quad x \in \mathbf{R}. \tag{2.4}$$

In performing the above transformation, a technical difficulty arises associated with the fact that the initial data w_0, v_0 are only assumed to be in $L^\infty \cap L^2$. Experience with parabolic equations suggests that we should not, in general, expect $v_{xx}(x,\cdot)$ and $v_t(x,\cdot)$

to be integrable on $[0, t]$ for solutions of (2.4). Thus we need to make (2.3) precise. It turns out that (2.3) holds in a weak sense. Moreover, (1.8) is equivalent to (2.4), and furthermore, (1.1) is equivalent to (1.3) for weak solutions. Specifically, we prove

Proposition 2.1. *Let $w_0(x), v_0(x) \in L^\infty(\mathbf{R}) \cap L^2(\mathbf{R})$, $k(t) \in C^1[0,\infty)$.*

(a) The initial value problems (1.1) and (1.3) are equivalent for weak solutions w, v of class $L^\infty([0,T]; L^\infty(\mathbf{R}) \cap L^2(\mathbf{R}))$.

(b) The initial value problems (1.8) and (2.4) are equivalent for weak solutions w, v of class $L^\infty([0,T]; L^\infty(\mathbf{R}) \cap L^2(\mathbf{R})) \cap L^2([0,T]; H^1(\mathbf{R}))$.

PROOF. We first consider part (a). Let $\{w(x,t), v(x,t)\}$ be bounded measurable functions satisfying (2.1) and (2.2). We proceed to show that $\{w(x,t),\ v(x,t)\}$ is a weak solution of (1.3); i.e., it satisfies (2.1) and

$$\int_0^T \int_{-\infty}^{\infty} [\eta_t v - \eta_x \varphi(w) + \eta \mathcal{F}[v]] dx dt + \int_{-\infty}^{\infty} \eta(x,0) v_0(x) dx = 0 \tag{2.5}$$

for any pair $\{\xi(x,t), \eta(x,t)\}$ of C^1 functions with compact support on $\mathbf{R}\times[0,T]$, satisfying $\xi(x,T) = \eta(x,T) = 0$.

Indeed, given any test function $\eta(x,t)$ we define

$$\zeta(x,t) = \eta(x,t) - \int_t^T r(\tau - t)\eta(x,\tau) d\tau. \tag{2.6}$$

Then $\zeta(x,t)$ is a C^1 function with compact support on $\mathbf{R}\times[0,T]$ and $\zeta(x,T) = 0$. Thus, $\zeta(x,t)$ can be used as a test function in (2.2). With $\zeta(x,t)$ defined by (2.6), an easy computation using (r) yields

$$\begin{aligned} &\int_0^T \int_{-\infty}^{\infty} \zeta_x (\varphi(w) + \int_0^t k(t-\tau)\varphi(w(\cdot,\tau))d\tau) dx dt \\ &\qquad = \int_0^T \int_{-\infty}^{\infty} \eta_x \varphi(w) dx dt, \end{aligned} \tag{2.7}$$

and, by (1.4),

$$\begin{aligned} &\int_0^T \int_{-\infty}^{\infty} \zeta_t v \, dx dt + \int_{-\infty}^{\infty} \zeta(x,0) v_0(x) dx = \\ &\qquad = \int_0^T \int_{-\infty}^{\infty} \Big(\eta_t + r(0)\eta + \int_t^T r'(\tau - t)\eta(\cdot,\tau)d\tau\Big) v \, dx dt \\ &\qquad + \int_{-\infty}^{\infty} \Big(\eta(x,0) - \int_0^T r(\tau)\eta(x,\tau) d\tau\Big) v_0(x) dx \\ &\qquad = \int_0^T \int_{-\infty}^{\infty} (\eta_t v + \eta \mathcal{F}[v]) dx dt + \int_{-\infty}^{\infty} \eta(x,0) v_0(x) dx. \end{aligned} \tag{2.8}$$

Combining (2.2) with (2.7) and (2.8) we conclude that $\{w(x,t),v(x,t)\}$ satisfies (2.5) and is thus a weak solution of (1.3).

Conversely, suppose that $\{w(x,t),v(x,t)\}$ satisfies (2.1) and (2.5). Given $\zeta(x,t)$, we solve the integral equation (2.6) for $\eta(x,t)$ and obtain

$$\eta(x,t)=\zeta(x,t)+\int_t^T k(\tau-t)\zeta(x,\tau)d\tau. \tag{2.9}$$

Then, we employ $\eta(x,t)$ as a test function in (2.5) and use (2.7) and (2.8) to arrive at (2.2). This concludes the proof of part (a).

By comparing the relations (2.2) and (2.5) we deduce that

$$\int_0^T\int_{-\infty}^{\infty}\Big[\zeta\mathcal{F}[v]+\zeta_x\int_0^t k(t-\tau)\varphi(w(\cdot,\tau))d\tau\Big]dxdt=0, \tag{2.10}$$

for every test function $\zeta(x,t)$. This is a weak form of relation (1.2).

The proof of part (b) follows the same pattern and will be omitted. Moreover, the analogue of (2.10) is the weak form of (2.3).

3. A PRIORI ESTIMATES.

Our object is to obtain a-priori estimates, independent of ε, for solutions $\{w^\varepsilon,v^\varepsilon\}$ of (1.8); we employ the equivalent system (2.4). For the remainder of this section we drop the superscript ε.

Let $\{w(x,t),v(x,t)\}$ be a solution of (2.4), corresponding to initial data w_0,v_0 in $L^\infty\cap L^2$, and tending to $(0,0)$ as $|x|\to\infty$. The solution is assumed sufficiently smooth to justify the steps in the derivation of the a-priori estimates (e.g. the regularity properties in Theorem 3.1 below suffice). In the sequel C will stand for a generic constant depending on the $L^\infty(\mathbf{R})$ and $L^2(\mathbf{R})$-norms of the initial data, on the $C^1[0,T]$-norm of $r(t)$, on properties of the function $\varphi(w)$, on T but not on ε. Whenever the constant depends on ε it will be denoted by C_ε.

Our analysis of (2.4) will be based on the concept of entropy-entropy flux pairs for the elastic problem (E) (cf. Lax [11]). A smooth, convex function $\eta(w,v)$ defined on $\mathbf{R}\times\mathbf{R}$ is an entropy for (E), with corresponding entropy flux $q(w,v)$, if

$$\partial_t\eta(w(x,t),v(x,t))+\partial_x q(w(x,t),v(x,t))=0 \tag{3.1}$$

for any smooth solution $\{w(x,t),v(x,t)\}$ of (E). Such pairs are generated as solutions of the system of equations

$$\begin{cases} q_w=-\varphi'(w)\eta_v\\ q_v=-\eta_w,\end{cases} \tag{3.2}$$

provided $\eta(w,v)$ is convex. Eliminating $q(w,v)$ in (3.2), we find that $\eta(w,v)$ must be a convex solution of the linear wave equation

$$\eta_{ww}=\varphi'(w)\eta_{vv}; \tag{3.3}$$

$q(w,v)$ is then determined by (3.2). A classical example of an entropy-entropy flux pair is

$$\eta(w,v) = \frac{1}{2}v^2 + \int_0^w \varphi(\xi)d\xi,\ q(w,v) = -v\varphi(w). \tag{3.4}$$

For each entropy-entropy flux pair $\{\eta(w,v), q(w,v)\}$ the functions $\{\bar\eta(w,v), \bar q(w,v)\}$ defined by

$$\begin{cases} \bar\eta(w,v) := \eta(w,v) - \eta(0,0) - \eta_w(0,0)w - \eta_v(0,0)v, \\ \bar q(w,v) := q(w,v) - q(0,0) + \eta_w(0,0)v - \eta_v(0,0)(\varphi(w) - \varphi(0)), \end{cases} \tag{3.5}$$

form an entropy-entropy flux pair for (E) that vanishes to quadratic order in (w,v) at $(0,0)$. Along solutions $\{w(x,t), v(x,t)\}$ of (2.4), a simple computation yields the identity

$$\begin{aligned} \partial_t\bar\eta(w,v) + \partial_x\bar q(w,v) &= \bar\eta_v(w,v)\mathcal{F}[v] + \varepsilon\partial_x^2\bar\eta(w,v) \\ &\quad - \varepsilon[\eta_{ww}(w,v)w_x^2 + 2\eta_{wv}(w,v)w_xv_x + \eta_{vv}(w,v)v_x^2]. \end{aligned} \tag{3.6}$$

Integrating (3.6) over $\mathbf{R}\times(0,t)$, $0 < t \le T$, we obtain

$$\begin{aligned} &\int_{-\infty}^{\infty} \bar\eta(w(x,t), v(x,t))dx \\ &\quad + \varepsilon\int_0^t\int_{-\infty}^{\infty} (\eta_{ww}(w,v)w_x^2 + 2\eta_{wv}(w,v)w_xv_x + \eta_{vv}(w,v)v_x^2) \\ &= \int_{-\infty}^{\infty} \bar\eta(w_0(x), v_0(x))dx + \int_0^t\int_{-\infty}^{\infty} \bar\eta_v(w,v)\mathcal{F}[v]dxd\tau, \quad 0 \le t \le T. \end{aligned} \tag{3.7}$$

For the first estimate, we employ the pair (3.4) and use (3.5) and (3.7) to deduce

$$\begin{aligned} &\int_{-\infty}^{\infty} \left[\frac{1}{2}v^2(x,t) + \Phi(w(x,t))\right]dx + \varepsilon\int_0^t\int_{-\infty}^{\infty} (\varphi'(w)w_x^2 + v_x^2)dxd\tau \\ &\quad = \int_{-\infty}^{\infty} \left[\frac{1}{2}v_0^2(x) + \Phi(w_0(x))\right]dx \\ &\qquad + \int_0^t\int_{-\infty}^{\infty} v\mathcal{F}[v]dxd\tau,\ 0 \le t \le T, \end{aligned} \tag{3.8}$$

where

$$\Phi(w) := \int_0^w (\varphi(\xi) - \varphi(0))d\xi. \tag{3.9}$$

On account of (1.5),

$$\varphi'(w) \ge \varphi'(w_i) > 0, \text{ for every } w \in \mathbf{R} \tag{3.10}$$

and

$$\Phi(w) \geq \frac{\varphi'(w_i)}{2} w^2 \geq 0, \text{ for every } w \in \mathbf{R}. \tag{3.11}$$

Using (1.4) and the Cauchy-Schwarz inequality,

$$\begin{aligned}
&\int_0^t \int_{-\infty}^{\infty} v\mathcal{F}[v]\,dx\,d\tau = \\
&= \int_0^t \int_{-\infty}^{\infty} v(x,\tau)\Big[r(0)v(x,\tau) - r(\tau)v_0(x) + \int_0^{\tau} r'(\tau - s)v(x,s)ds\Big]dx\,d\tau \\
&\leq C(1+T)\max_{[0,T]}\{|r(t)| + |r'(t)|\}\Big(\int_{-\infty}^{\infty} v_0^2(x)dx + \int_0^t \int_{-\infty}^{\infty} v^2 dx\,d\tau\Big) \\
&\leq C\Big(1 + \int_0^t \int_{-\infty}^{\infty} v^2(x,\tau)dx\,d\tau\Big), \quad 0 \leq t \leq T.
\end{aligned} \tag{3.12}$$

Combining (3.8) with (3.10) and (3.12)

$$\int_{-\infty}^{\infty} v^2(x,t)dx \leq C + C\int_0^t \int_{-\infty}^{\infty} v^2 dx\,d\tau, \tag{3.13}$$

whence by the Gronwall inequality, (3.8), (3.10) and (3.11),

$$\begin{aligned}
&\int_{-\infty}^{\infty} [v^2(x,t) + w^2(x,t)]dx \\
&\qquad + \varepsilon\int_0^t \int_{-\infty}^{\infty} (w_x^2 + v_x^2)dx\,d\tau \leq C, \quad 0 \leq t \leq T,
\end{aligned} \tag{3.14}$$

where C is independent of ε.

Our next objective is to establish a-priori L^∞ estimates, independent of ε, for solutions $\{w(x,t), v(x,t)\}$ of (2.4). We follow the development of Dafermos [3]. The following facts are proved in [3, Section 2]: For $\varphi(w)$ as in (1.5), the wave equation (3.3) admits a class of solutions $\{\eta^{(k)}(w,v)\}_{k>0}$ on $\mathbf{R}\times\mathbf{R}$ which are strictly convex and grow exponentially at infinity. These solutions have the form

$$\eta^{(k)}(w,v) = Y^{(k)}(w)\cosh kv, \quad 0 < k < \infty, \tag{3.15}$$

where $Y^{(k)}(w)$ is the solution of the initial value problem

$$\begin{cases} Y^{(k)''}(w) = k^2\varphi'(w)Y^{(k)}(w), \\ Y^{(k)}(w_i) = 1, \ Y^{(k)'}(w_i) = 0, \quad 0 < k < \infty. \end{cases} \tag{3.16}$$

The functions $Y^{(k)}(w)$ satisfy the estimates

$$Y^{(k)}(w) \geq \cosh[k\sqrt{\varphi'(w_i)}(w - w_i)], \quad -\infty < w < \infty,\ 0 < k < \infty, \tag{3.17}$$

$$|Y^{(k)\prime}(w)| \leq k\sqrt{\varphi'(w)}Y^{(k)}(w), \quad -\infty < w < \infty,\ 0 < k < \infty \tag{3.18}$$

and

$$Y^{(k)}(w) \leq \exp[k\int_{w_i}^{w} \sqrt{\varphi'(\xi)}d\xi], \quad -\infty < w < \infty,\ 0 < k < \infty. \tag{3.19}$$

We will estimate the solution $\{w(x,t), v(x,t)\}$ of (2.4) by monitoring the evolution of

$$\int_{-\infty}^{\infty} \bar{\eta}^{(k)}(w(x,t), v(x,t))dx.$$

In view of the convexity of $\bar{\eta}^{(k)}(w,v)$, (3.7) yields

$$\begin{aligned}\int_{-\infty}^{\infty} \bar{\eta}^{(k)}(w(x,t), v(x,t))dx &\leq \int_{-\infty}^{\infty} \bar{\eta}^{(k)}(w_0(x), v_0(x))dx \\ &+ \int_0^t \int_{-\infty}^{\infty} \bar{\eta}_v^{(k)}(w(x,\tau), v(x,\tau))\mathcal{F}[v](x,\tau)dxd\tau, \quad 0 \leq t \leq T,\end{aligned} \tag{3.20}$$

where, by (3.5), (3.15),

$$\bar{\eta}^{(k)}(w,v) = \eta^{(k)}(w,v) - Y^{(k)}(0) - Y^{(k)\prime}(0)w \geq 0, \tag{3.21}$$

and

$$\bar{\eta}_v^{(k)}(w,v) = k(\tanh\ kv)\eta^{(k)}(w,v). \tag{3.22}$$

Using (3.22), (3.21), (3.18), the Cauchy-Schwarz inequality and (1.4) the last integral in (3.20) can be estimated as follows:

$$\begin{aligned}&\left|\int_0^t \int_{-\infty}^{\infty} \bar{\eta}_v^{(k)}(w,v)\mathcal{F}[v]dxd\tau\right| \\ &= \left|\int_0^t \int_{-\infty}^{\infty} k(\tanh\ kv)\Big(\bar{\eta}^{(k)}(w,v) + Y^{(k)}(0) \right. \\ &\qquad \left. + Y^{(k)\prime}(0)w\Big)\mathcal{F}[v]dxd\tau\right| \\ &\leq Ck^2Y^{(k)}(0)\int_0^t \int_{-\infty}^{\infty} (|w| + |v|)|\mathcal{F}[v]|dxd\tau \\ &\qquad + k\int_0^t \int_{-\infty}^{\infty} \bar{\eta}^{(k)}(w,v)|\mathcal{F}[v]|dxd\tau \\ &\leq Ck^2Y^{(k)}(0)[\int_0^t \int_{-\infty}^{\infty} (w^2 + v^2)dxd\tau + \int_{-\infty}^{\infty} v_0^2(x)dx] \\ &\qquad + Ck\int_0^t M(\tau)[\int_{-\infty}^{\infty} \bar{\eta}^{(k)}(w(x,\tau), v(x,\tau))dx]d\tau,\end{aligned} \tag{3.23}$$

where

$$M(t) := \sup_{x\in\mathbf{R}}\{1 + |v(x,t)| + \int_0^t |v(x,\tau)|d\tau\}, \quad 0 \le t \le T. \tag{3.24}$$

Combining (3.20), (3.23) and (3.14), we find

$$\int_{-\infty}^{\infty} \bar\eta^{(k)}(w(x,t),v(x,t))dx \le \int_{-\infty}^{\infty} \bar\eta^{(k)}(w_0(x),v_0(x))dx + Ck^2 Y^{(k)}(0)$$

$$+ Ck\int_0^t M(\tau)\Big\{\int_{-\infty}^{\infty} \bar\eta^{(k)}(w(x,\tau),v(x,\tau))dx\Big\}d\tau, \quad 0 \le t \le T. \tag{3.25}$$

Then the Gronwall inequality yields the estimate

$$\int_{-\infty}^{\infty} \bar\eta^{(k)}(w(x,t),v(x,t))dx$$

$$\le \Big\{\int_{-\infty}^{\infty} \bar\eta^{(k)}(w_0(x),v_0(x))dx$$

$$+ Ck^2 Y^{(k)}(0)\Big\}\exp\Big(Ck\int_0^t M(\tau)d\tau\Big), \quad 0 \le t \le T. \tag{3.26}$$

Our next objective is to take the $\frac{1}{k}$ power of relation (3.26) and pass to the limit $k \to \infty$, in order to obtain a relation involving the L^∞-norms of w and v. To this end, we first estimate the right-hand side of (3.26). If $\lambda^{(k)}_{max}(w,v)$ is the largest eigenvalue of the Hessian of the convex function $\bar\eta^{(k)}(w,v)$, then

$$\bar\eta^{(k)}(w,v) \le \frac{1}{2}\big(\max_{w^2+v^2\le 1} \lambda^{(k)}_{max}(w,v)\big)(w^2+v^2)$$

$$\le \frac{1}{2}\big[\max_{w^2+v^2\le 1} (\eta^{(k)}_{ww}(w,v) + \eta^{(k)}_{vv}(w,v))\big](w^2+v^2), \quad w^2+v^2 \le 1; \tag{3.27}$$

whence, using (3.15), (3.16) and (3.19), we deduce

$$\bar\eta^{(k)}(w,v) \le Ck^2 e^{Ck}(w^2+v^2), \quad w^2+v^2 \le 1. \tag{3.28}$$

Set $\mathcal{O} := \{x \in \mathbf{R} : w_0^2(x) + v_0^2(x) > 1\}$, and observe that $\mathcal{O}$ has finite Lebesgue measure, $0 \le |\mathcal{O}| < \infty$. Then, using (3.28), (3.21), (3.18), (3.15) and (3.19) we arrive at

$$\int_{-\infty}^{\infty} \bar\eta^{(k)}(w_0(x),v_0(x))dx = \int_{R\setminus\mathcal{O}} \bar\eta^{(k)}(w_0(x),v_0(x))dx$$

$$+ \int_{\mathcal{O}} \bar\eta^{(k)}(w_0(x),v_0(x))dx$$

$$\le Ck^2 e^{Ck}\int_{-\infty}^{\infty}(w_0^2(x)+v_0^2(x))dx$$

$$+ C|\mathcal{O}|(1+k)Y^{(k)}(0)\sup_{x\in\mathbf{R}}|w_0(x)|$$

$$+ |\mathcal{O}|\exp\Big\{k\sup_{x\in\mathbf{R}}\Big[|v_0(x)| + \Big|\int_{w_*}^{w_0(x)} \sqrt{\varphi'(\xi)}d\xi\Big|\Big]\Big\}. \tag{3.29}$$

Finally, combining (3.26) with (3.29) and (3.19),

$$\int_{-\infty}^{\infty} \bar{\eta}^{(k)}(w(x,t),v(x,t))dx \le C(k^2+k+1)\exp\{k(A+\int_0^t M(\tau)d\tau)\}, \tag{3.30}$$

where the constant A only depends on the L^∞-norm of the initial data and the values of $\varphi(w)$ and $\varphi'(w)$ on the smallest interval where $w_0(x)$ and w_i take values. Next, we estimate the left-hand side of (3.30). We set

$$S(t) := \sup_{x\in\mathbf{R}}\{1+|w(x,t)|+|v(x,t)|\} \tag{3.31}$$

and consider the set

$$E_\delta(t) := \{x \in R : S(t)-\delta \le 1+|w(x,t)|+|v(x,t)| \le S(t)\},$$

where $\delta > 0$ small. Then $|E_\delta(t)| < \infty$. Moreover, by virtue of (3.21), (3.18), (3.15) and (3.17) we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} \bar{\eta}^{(k)}(w(x,t),v(x,t))dx &\ge \int_{E_\delta(t)} \bar{\eta}^{(k)}(w(x,t),v(x,t))dx \\ &\ge \int_{E_\delta(t)} \eta^{(k)}(w(x,t),v(x,t))dx \\ &\quad - C(1+k)Y^{(k)}(0)\int_{E_\delta(t)}(1+|w(x,t)|)dx \\ &\ge \frac{|E_\delta(t)|}{C}e^{\frac{k}{C}(S(t)-\delta)-Ck} - C(1+k)Y^{(k)}(0)|E_\delta(t)|S(t).\end{aligned} \tag{3.32}$$

We combine (3.29) with (3.30) and raise the resulting inequality to the $1/k$ power. Then, we first let $k \to \infty$ and then $\delta \downarrow 0$. After taking the logarithm of both sides we conclude

$$\sup_{x\in R}\{1+|w(x,t)|+|v(x,t)|\} \le C + C\int_0^t M(\tau)d\tau, \quad 0 \le t \le T. \tag{3.33}$$

Substituting (3.24) into (3.33) and using (3.31) yields

$$S(t) \le C + C(1+T)\int_0^t S(\tau)d\tau. \tag{3.34}$$

Finally, integrating (3.34) we conclude

$$\sup_{x\in\mathbf{R}}\{1+|w(x,t)|+|v(x,t)|\} \le M, \quad 0 \le t \le T, \tag{3.35}$$

where the constant M depends on T but not on ε. Therefore, the solution $\{w(x,t), v(x,t)\}$ of (2.4) is bounded on $[0,T] \times \mathbf{R}$, uniformly in $\varepsilon > 0$.

The a priori estimates (3.14) and (3.35) together with an existence theorem for the regularized problem (1.8) are summarized below.

Theorem 3.1. *Under the hypotheses (1.5)-(1.7), for each $\varepsilon > 0$, $T > 0$, the initial value problem (2.4) (respectively (1.8)) has a unique solution $\{w(x,t), v(x,t)\}$ defined on $R \times [0,T]$ such that $w, v \in C([0,T]; L^2(\mathbf{R})) \cap L^\infty(\mathbf{R} \times [0,T])$, $w_x, v_x, w_{xx}, v_{xx}, w_t, v_t \in C((0,T]; L^2(\mathbf{R}))$ and, also, $w_x, v_x \in L^2(\mathbf{R} \times [0,T])$. In addition, as $\varepsilon \searrow 0$, the families $\{w(x,t), v(x,t)\}_{\varepsilon>0}$ and $\{\varepsilon^{1/2} w_x(x,t), \varepsilon^{1/2} v_x(x,t)\}_{\varepsilon>0}$ lie in bounded sets of $L^\infty([0,T]; L^2(\mathbf{R}) \cap L^\infty(\mathbf{R}))$ and $L^2(\mathbf{R} \times [0,T])$, respectively.*

PROOF. It remains to prove the existence part of Theorem 2.1.

In view of (3.35) we modify the function φ as follows. Let M be the a-priori L^∞-bound in (3.35) and choose $\ell > max\{M, |w_i|\}$. We modify the function $\varphi(w)$ outside the interval $[-\ell, \ell]$ so that the resulting function $\tilde{\varphi}(w)$ is twice continuously differentiable and satisfies

$$\tilde{\varphi}(w) := \begin{cases} \varphi(w) \text{ if } |w| \leq \ell \\ \text{a linear function if } |w| > 2\ell \end{cases} \tag{3.36}$$

and

$$\varphi'(w_i) \leq \tilde{\varphi}'(w) \leq 1 + \max_{-\ell \leq w \leq \ell} \varphi'(w) =: L, \quad w \in \mathbf{R}. \tag{3.37}$$

Then the derivation of (3.35) suggests that any solution $\{w(x,t), v(x,t)\}$ of the system (2.4) with φ replaced by $\tilde{\varphi}$ will satisfy the bound (3.35) with the same constant M. (We note here that, although the family of entropies for the modified system (2.4) differs from the original (cf. (3.15), (3.16)), the estimates (3.17), (3.18), (3.19), (3.29) and (3.32) hold with the same constants for both families). By virtue of the above remarks we may, without loss of generality, assume that the function $\varphi(w)$ in (2.4) has been modified to comply with (3.36) and (3.37).

Existence and uniqueness of solutions for (2.4) on $\bar{Q}_T = \mathbf{R} \times [0,T]$ will be deduced by a simple application of the contraction mapping theorem. Let $\mathcal{B}$ be the Banach space

$$\begin{aligned} \mathcal{B} := \{(w,v) : \bar{Q}_{T^*} \to \mathbf{R} \times \mathbf{R}, w, v \in L^\infty([0,T^*]; L^2(\mathbf{R})), \\ w_x, v_x \in L^2([0,T^*]; L^2(\mathbf{R}))\} \end{aligned}$$

for some $0 < T^* \leq T$, with norm

$$|||(w,v)|||^2_{\mathcal{B}} := \sup_{0 \leq t \leq T^*} \int_{-\infty}^{\infty} [w^2(x,t) + v^2(x,t)] dx + \varepsilon \int_0^{T^*} \int_{-\infty}^{\infty} (w_x^2 + v_x^2) dx dt.$$

Let S be the map that carries $(W, V) \in \mathcal{B}$ into the solutions (w, v) of the linear parabolic initial value problem:

$$\begin{cases} w_t = V_x + \varepsilon w_{xx} \\ v_t = \varphi(W)_x + \mathcal{F}[V] + \varepsilon v_{xx} \\ w(x,0) = w_0(x), \; v(x,0) = v_0(x). \end{cases} \tag{3.38}$$

By standard theory of the heat equation $S : \mathcal{B} \to \mathcal{B}$, for any $T^* > 0$. We proceed to show that, for T^* small enough, S is a contraction.

Let $(W_1, V_1), (W_2, V_2) \in \mathcal{B}$ and let $(w_1, v_1), (w_2, v_2)$ be the respective solutions of (3.38) corresponding to the same initial data $w_i(x, 0) = w_0(x)$, $v_i(x, 0) = v_0(x)$, $i = 1, 2$. Simple energy estimates for (3.38) yield

$$\begin{aligned}
&\int_{-\infty}^{\infty} (w_1(x,t) - w_2(x,t))^2 dx + \varepsilon \int_0^t \int_{-\infty}^{\infty} (w_1(x,\tau) - w_2(x,\tau))_x^2 dx d\tau \\
&\qquad \le \frac{1}{\varepsilon} \int_0^t \int_{-\infty}^{\infty} (V_1(x,\tau) - V_2(x,\tau))^2 dx d\tau,
\end{aligned} \tag{3.39}$$

$$\begin{aligned}
&\int_{-\infty}^{\infty} (v_1(x,t) - v_2(x,t))^2 dx + \varepsilon \int_0^t \int_{-\infty}^{\infty} (v_1(x,\tau) - v_2(x,\tau))_x^2 dx d\tau \\
&\qquad \le \frac{1}{\varepsilon} \int_0^t \int_{-\infty}^{\infty} (\varphi(W_1(x,\tau)) - \varphi(W_2(x,\tau)))^2 dx d\tau \\
&\qquad + \int_0^t \int_{-\infty}^{\infty} (v_1(x,\tau) - v_2(x,\tau))^2 dx d\tau \\
&\qquad + \int_0^t \int_{-\infty}^{\infty} (\mathcal{F}[V_1(x,\tau) - V_2(x,\tau)])^2 dx d\tau.
\end{aligned} \tag{3.40}$$

Moreover, in virtue of (3.37)

$$\begin{aligned}
&\int_0^t \int_{-\infty}^{\infty} (\varphi(W_1(x,\tau)) - \varphi(W_2(x,\tau)))^2 dx d\tau \\
&\qquad \le L^2 \int_0^t \int_{-\infty}^{\infty} (W_1(x,\tau) - W_2(x,\tau))^2 dx d\tau
\end{aligned} \tag{3.41}$$

and by (1.4)

$$\begin{aligned}
&\int_0^t \int_{-\infty}^{\infty} (\mathcal{F}[V_1(x,\tau) - V_2(x,\tau)])^2 dx d\tau \\
&\qquad \le 2\big[\max_{[0,T]}(|r(t)| + |r'(t)|)^2\big](T^2 + 1) \int_0^t \int_{-\infty}^{\infty} (V_1(x,\tau) - V_2(x,\tau))^2 dx d\tau.
\end{aligned} \tag{3.42}$$

Combining (3.39)-(3.42) and using the definition of norm in $\mathcal{B}$, one obtains

$$\begin{aligned}
&|||((w_1 - w_2), (v_1 - v_2))|||_{\mathcal{B}}^2(t) \le C_\varepsilon T^* |||((W_1 - W_2), (V_1 - V_2))|||_{\mathcal{B}}^2(T^*) \\
&\qquad + \int_0^t |||((w_1 - w_2), (v_1 - v_2))|||_{\mathcal{B}}^2(\tau) d\tau,
\end{aligned} \tag{3.43}$$

where $C_\epsilon > 0$ is a constant depending on $\frac{1}{\epsilon}$, L, T, and $\|R\|_{C^1[0,T]}$. By the Gronwall inequality,

$$
\begin{aligned}
&|||((w_1 - w_2), (v_1 - v_2))|||_{\mathcal{B}}^2(T^*) \\
&\leq C_\epsilon T^* |||((W_1 - W_2), (V_1 - V_2))|||_{\mathcal{B}}^2(T^*),
\end{aligned}
\tag{3.44}
$$

and $\mathcal{S}$ is a strict contraction on $\mathcal{B}$, provided T^* is chosen sufficiently small.

By proceeding in steps of length T^*, the above procedure yields a unique solution $\{w(x,t), v(x,t)\}$ on $\mathbf{R} \times [0,T]$ of the system (2.4), for each fixed $\epsilon > 0$ and any $T > 0$. By standard L^2-theory of the heat equation [10], the solution $w(x,t), v(x,t) \in C([0,T]; L^2(\mathbf{R}))$ and has the regularity

$$
\sqrt{t}\, w_x,\ \sqrt{t}\, v_x \in L^\infty([0,T]; L^2(\mathbf{R})),
\tag{3.45}
$$

$$
w_x, v_x, \sqrt{t}\, w_t, \sqrt{t}\, v_t, \sqrt{t}\, w_{xx}, \sqrt{t}\, v_{xx} \in L^2(\mathbf{R} \times [0,T]).
\tag{3.46}
$$

The solution $\{w, v\}$ enjoys additional regularity which is obtained as follows. Differentiate $(2.4)_1$ and $(2.4)_2$ with respect to t (the rigorous justification is by taking difference quotients) and use (1.4) to obtain

$$
w_{tt} = v_{xt} + \epsilon w_{xxt},
\tag{3.47}
$$

$$
v_{tt} = \varphi(w)_{xt} + r(0) v_t(x,t) + \int_0^t r'(t-\tau) v_t(x,\tau) d\tau + \epsilon v_{xxt}.
\tag{3.48}
$$

Multiply (3.47) by $t^2 w_t$ and integrate over $\mathbf{R} \times [0,T]$. Integration by parts, (3.35), (3.45), (3.46) yield

$$
\begin{aligned}
&t^2 \int_{-\infty}^{\infty} w_t^2(x,t) dx + \int_0^t \int_{-\infty}^{\infty} \tau^2 w_{xt}^2 dx d\tau \\
&\leq C_\epsilon \int_0^t \int_{-\infty}^{\infty} (\tau v_t^2 + \tau w_t^2) dx d\tau \leq C_\epsilon.
\end{aligned}
\tag{3.49}
$$

Similarly, multiplying (3.48) by $t^2 v_t$ and performing the same steps, we obtain

$$
\begin{aligned}
&t^2 \int_{-\infty}^{\infty} v_t^2(x,t) dx + \int_0^t \int_{-\infty}^{\infty} \tau^2 v_{xt}^2 dx d\tau \\
&\leq C_\epsilon \int_0^t \int_{-\infty}^{\infty} (\tau v_t^2 + \tau w_t^2) dx d\tau \leq C_\epsilon.
\end{aligned}
\tag{3.50}
$$

Next, we multiply (3.47) and (3.48) by $t^3 w_{tt}$ and $t^3 v_{tt}$, respectively, integrate by parts over $\mathbf{R} \times [0,t]$ and use (3.35), (3.45), (3.46), (3.49), (3.50) and the Cauchy-Schwarz inequality to arrive at

$$\int_0^t \int_{-\infty}^{\infty} \tau^3 w_{tt}^2 dx d\tau + t^3 \int_{-\infty}^{\infty} w_{xt}^2(x,t) dx$$

$$\leq C_\varepsilon \int_0^t \int_{-\infty}^{\infty} (\tau^2 v_{xt}^2 + \tau^2 w_{xt}^2) dx d\tau \leq C_\varepsilon, \tag{3.51}$$

and

$$\int_0^t \int_{-\infty}^{\infty} \tau^3 v_{tt}^2 dx d\tau + t^3 \int_{-\infty}^{\infty} v_{xt}^2(x,t) dx$$

$$\leq C_\varepsilon \int_0^t \int_{-\infty}^{\infty} (\tau^2 v_{xt}^2 + \tau^3 (\varphi(w)_{xt})^2 + \tau v_t^2) dx d\tau$$

$$\leq C_\varepsilon (1 + \int_0^t \int_{-\infty}^{\infty} \tau^3 w_x^2 w_t^2 dx d\tau)$$

$$\leq C_\varepsilon (1 + \int_0^t \tau \sup_x w_x^2(x,\tau) d\tau)$$

$$\leq C_\varepsilon (1 + \int_0^t \tau \int_{-\infty}^{\infty} (w_x^2 + w_{xx}^2) dx d\tau) \leq C_\varepsilon. \tag{3.52}$$

Finally, using (3.45), (3.46), (3.49), (3.50), (3.51), (3.52) and standard embedding theorems we conclude that $\{w(x,t), v(x,t)\}$ has the regularity claimed in Theorem 3.1.

4. THE PASSAGE TO THE LIMIT $\varepsilon \searrow 0$.

For $\varepsilon > 0$, let $\{w^\varepsilon(x,t), v^\varepsilon(x,t)\}$ be the solution of the initial value problem (1.8) on $Q_T := \mathbf{R} \times [0,T]$, with regularity properties as in Theorem 3.1. By virtue of (3.35) the family of functions $\{w^\varepsilon(x,t), v^\varepsilon(x,t)\}_{\varepsilon>0}$ is uniformly bounded in $L^\infty(Q_T)$. There exist functions $w(x,t), v(x,t)$ and $\bar{\varphi}(x,t)$ in $L^\infty(Q_T)$ such that, along a subsequence,

$$w^\varepsilon(x,t) \overset{*}{\rightharpoonup} w(x,t) \text{ in } L^\infty - \text{weak star}, \tag{4.1}$$

$$v^\varepsilon(x,t) \overset{*}{\rightharpoonup} v(x,t) \text{ in } L^\infty - \text{weak star}, \tag{4.2}$$

and

$$\varphi(w^\varepsilon(x,t)) \overset{*}{\rightharpoonup} \bar{\varphi}(x,t) \text{ in } L^\infty - \text{weak star}, \tag{4.3}$$

as $\varepsilon \searrow 0$. The object of this section is to show that $\{w(x,t), v(x,t)\}$ is a solution of (1.1) in the sense of distributions.

The major obstacle to overcome is that, in general, nonlinear functions are not continuous under weak star convergence, and it does not follow that $\tilde{\varphi}(x,t) = \varphi(w(x,t))$. Under weak-star convergence, such composite weak limits have been characterized by Tartar [21, 22, 23] as expected values of a family of probability measures $\nu_{(x,t)}$, called the Young measures. The relevant issue here is whether the Young measure reduces to a Dirac mass. Using the theory of compensated compactness and a class of entropy-entropy flux pairs introduced by Lax [11], DiPerna [7] proved the following key result which will be employed in the sequel.

Proposition 4.1 (DiPerna). *Let $\{w^\varepsilon, v^\varepsilon\} : Q_T \to \mathbf{R}$ be a collection of functions such that*

$$\|w^\varepsilon\|_{L^\infty(Q_T)} + \|v^\varepsilon\|_{L^\infty(Q_T)} \le C,$$

where C is a constant independent of ε. Suppose also that for any smooth entropy-entropy flux pair $\{\eta(w,v), q(w,v)\}$ of (E) (see Section 3), with φ satisfying (1.5),

$$\partial_t \eta(w^\varepsilon(x,t), v^\varepsilon(x,t)) + \partial_x q(w^\varepsilon(x,t), v^\varepsilon(x,t))$$

lies in a compact set of $H^{-1}_{loc}(Q_T)$. Then there exists a subsequence $\{w^{\varepsilon'}, v^{\varepsilon'}\}$ and functions $w, v \in L^\infty(Q_T)$ such that

$$w^{\varepsilon'}(x,t) \to w(x,t),\ v^{\varepsilon'}(x,t) \to v(x,t), \text{ a.e. for } (x,t) \in Q_T$$

as $\varepsilon' \downarrow 0$.

We apply Proposition 4.1 to the family $\{w^\varepsilon(x,t), v^\varepsilon(x,t)\}_{\varepsilon>0}$ of solutions of (1.8) (which also satisfy (2.4)). A straightforward calculation using (2.4) and (3.2) yields

$$\begin{aligned}
&\partial_t \eta(w^\varepsilon, v^\varepsilon) + \partial_x q(w^\varepsilon, v^\varepsilon) \\
&\quad = \varepsilon^{1/2} \partial_x(\varepsilon^{1/2} \eta_w(w^\varepsilon, v^\varepsilon) w^\varepsilon_x + \varepsilon^{1/2} \eta_v(w^\varepsilon, v^\varepsilon) v^\varepsilon_x) \\
&\quad\quad - \varepsilon[\eta_{ww}(w^\varepsilon, v^\varepsilon)(w^\varepsilon_x)^2 + 2\eta_{wv}(w^\varepsilon, v^\varepsilon) w^\varepsilon_x v^\varepsilon_x + \eta_{vv}(w^\varepsilon, v^\varepsilon)(v^\varepsilon_x)^2] \\
&\quad\quad + \eta_v(w^\varepsilon, v^\varepsilon) \mathcal{F}[v^\varepsilon].
\end{aligned} \tag{4.4}$$

The a-priori estimates (3.14) and (3.35) imply that the family

$$\varepsilon^{1/2} \partial_x(\varepsilon^{1/2} \eta_w(w^\varepsilon, v^\varepsilon) w^\varepsilon_x + \varepsilon^{1/2} \eta_v(w^\varepsilon, v^\varepsilon) v^\varepsilon_x)$$

converges to 0 and is thereby compact in $H^{-1}(Q_T)$, while the family

$$\varepsilon[\eta_{ww}(w^\varepsilon, v^\varepsilon)(w^\varepsilon_x)^2 + 2\eta_{wv}(w^\varepsilon, v^\varepsilon) w^\varepsilon_x v^\varepsilon_x + \eta_{vv}(w^\varepsilon, v^\varepsilon)(v^\varepsilon_x)^2]$$

is bounded in $L^1(Q_T)$. In addition, the family

$$\partial_t \eta(w^\varepsilon, v^\varepsilon) + \partial_x q(w^\varepsilon, v^\varepsilon) - \eta_v(w^\varepsilon, v^\varepsilon) \mathcal{F}[v^\varepsilon] \tag{4.5}$$

resides in a bounded set of $W^{-1,\infty}(Q_T)$. Using a lemma of Murat [14] we deduce that the quantity (4.5) lies in a compact set of $H^{-1}_{loc}(Q_T)$. Finally, since $\eta_v(w^\varepsilon, v^\varepsilon)\mathcal{F}[v^\varepsilon]$ lies in a bounded set of $L^2(Q_T)$ and thus in a compact set of $H^{-1}_{loc}(Q_T)$, the family

$$\partial_t \eta(w^\varepsilon, v^\varepsilon) + \partial_x q(w^\varepsilon, v^\varepsilon)$$

also lies in a compact set of $H^{-1}_{loc}(Q_T)$. Thus, Proposition 4.1 implies that

$$w^{\varepsilon'}(x,t) \to w(x,t),\ v^{\varepsilon'}(x,t) \to v(x,t) \text{ a.e. in } Q_T,$$

along a subsequence $\varepsilon' \searrow 0$, and permits passage to the limit $\varepsilon' \searrow 0$ in (1.8) in the sense of distributions. The pair of functions $\{w(x,t), v(x,t)\}$ belongs to $L^\infty([0,T]; L^2(\mathbf{R})) \cap L^\infty(Q_T)$ and is a weak solution of (1.1) on Q_T. This completes the proof of Theorem 1.1.

Finally, we remark that one does not expect weak solutions of (1.3) to be unique. We note that for every (convex) entropy $\eta(w,v)$, the solution constructed here satisfies the inequality

$$\partial_t \eta(w,v) + \partial_x q(w,v) \leq \eta_v(w,v)\mathcal{F}[v]$$

in the sense of distributions.

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